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BEHAVIOR OF STRUCTURES SUBJECTED TO A FORCED VIBRATION

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BEHAVIOR OF STRUCTURES SUBJECTED TO A FORCED VIBRATION

Charles T. G. Looney,[1] A.M. ASCE

SYNOPSIS

This paper describes modes of free vibration and the reciprocal theorem and Muller-Breslau's principle as applied to a steady state forced vibration. A parallel is drawn between the behavior of a simply supported beam and a continuous structure whereby the similarity is shown between the shapes of the mode of free vibration, the influence line for deflection, and the influence line for bending moment. These simple and accurate relationships are basic to the understanding of the behavior of continuous structures subjected to a steady state forced vibration.

1. INTRODUCTION

In this study the behavior of some relatively simple continuous structures will be described. The results described, however, present a true picture of the behavior of any continuous structure. The method of analysis, described by the author in A.S.C.E. Trans., Vol. 118, 1953, p. 794 was used in this study. That analysis makes use of superposition; in addition, this paper introduces the reciprocal theorem and Muller – Breslau's principle with modifications necessary for their application to the study of vibrations.

The effect on a continuous structure of a periodic force $P \sin 2\pi ft$ will be considered only as it affects the steady state condition.

There are two important phenomena to consider with regard to the effect of a periodic force on the deflections, bending moments and shears of a structure. These are the frequencies of free vibration of the structure and the influence of the position of the periodic force. Considerable attention will be given to the determination of the frequencies of free vibration of a structure and the shape of the deflection curves. Frequencies of free vibration are important because it is at these frequencies that the deflection due to a periodic force becomes very large. The influence of the position of the periodic force can best be described by means of influence lines.

2. Frequencies of Free Vibration of Continuous Structures.

In general, free vibrations of continuous structures occur with periodic moments at the ends of the members. At any given frequency of free vibration the relative magnitude of the end moments is constant and the deflection is proportional to the end moments. If the structure is simply supported and all the individual members have the same natural frequency, then free vibrations can occur with no end moments. Each member will deflect in the shape of the mode for a simply supported beam for the particular frequency. A

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continuous beam of identical spans is simple to analyze and will serve to illustrate free vibrations. The general requirements are that the periodic end moments and the end slopes be equal at all times. The relative values of the end moments and the frequencies of free vibrations can be found by equations similar to slope deflection equations.

The equations for the deflection of a simply supported beam due to a periodic moment on the left end are —

$$y = \frac{2Ml^2}{\pi^3 EI} \sum_{n=1,2,3,\ldots} \frac{1}{n^3} \beta_n \sin \frac{n\pi x}{l} \sin 2\pi f t \tag{1}$$

where $\beta_n = \frac{1}{1 - f^2/f_n^2}$ and the end moment is $M \sin 2\pi f t$ (M positive clockwise) for no damping, and

$$y = \frac{2Ml^2}{\pi^3 EI} \sum_{n=1,2,3,\ldots} \frac{1}{n^3} \beta_n \sin \frac{n\pi x}{l} e^{-2\pi \delta t} \sin 2\pi f t \tag{2}$$

where $\beta_n = \frac{1}{1 - \frac{f^2 + \delta^2}{f_n^2}}$ and the moment is $M e^{-2\pi\delta t} \sin 2\pi f t$

for damping proportional to the velocity.

In Equation 2 the deflection and end moment decrease logarithmically from any given value. The deflections, end moments and frequencies of free vibration are only slightly affected by small amount of damping. In this analysis Equation 1 will be used. The effect of damping would be a constant decrease, in proportion to $e^{-2\pi\frac{\delta}{t}}$ for a length of time equal to the period $\frac{1}{f}$ of both the end moments and deflections.

From Equation 1 the end slopes can be written as,

$$\theta_{x=0} = \frac{dy}{dx} = (a)\frac{Ml}{EI} \sin 2\pi f t$$

and

$$\theta_{x=l} = \frac{dy}{dx} = (b)\frac{Ml}{EI} \sin 2\pi f t$$

The values of the coefficients (a) and (b)[2] depend on the ratio of the frequency, f, of the periodic end moment, to the fundamental frequency, f_1, of the member. Figs. 1 and 2 show values of (a) and (b) as ordinates with values of the ratio f/f_1 as abscissa.

For a continuous beam of two identical spans, simply supported, the following equations can be written at the center support. The supports will be numbered from left to right.

$$(a)\, M_{21} = (a)\, M_{23}$$

and

$$M_{21} + M_{23} = 0$$

2. Expressions for coefficients (a) and (b) on page 8.

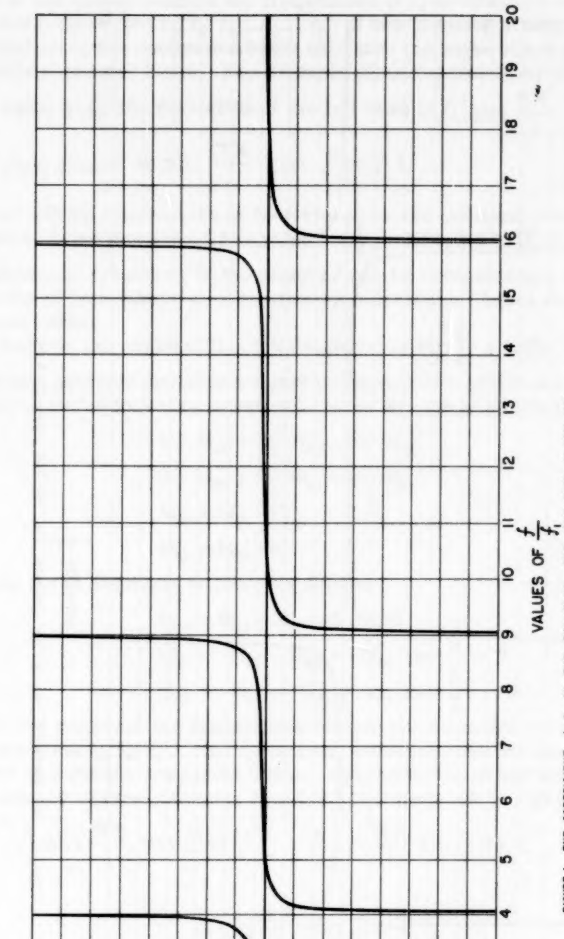


FIGURE 1. THE COEFFICIENT (a) IN THE EQUATION FOR THE END SLOPE AT x = 0

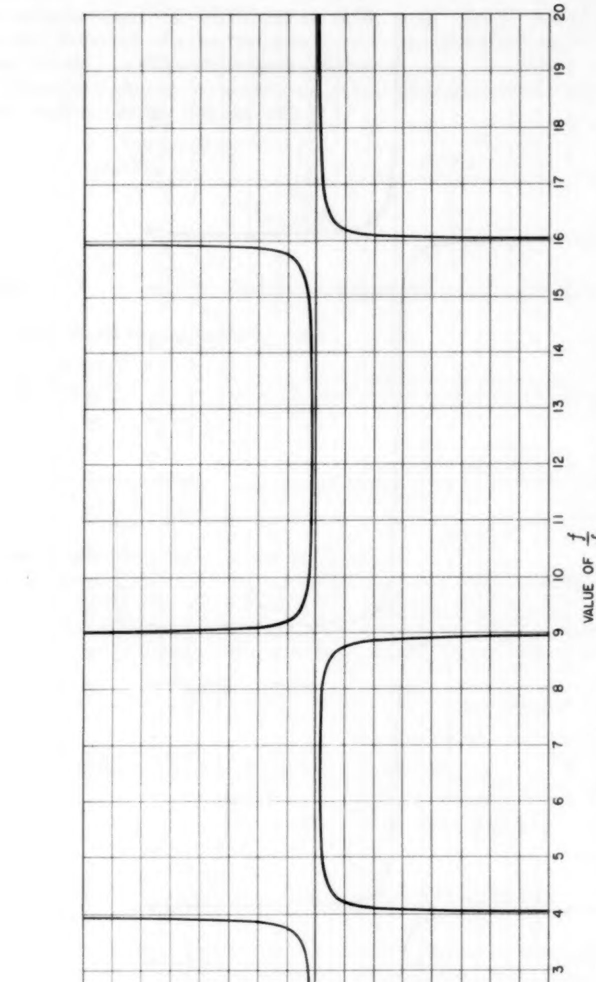


FIGURE 2. THE COEFFICIENT (b) IN THE EQUATION FOR THE END SLOPE AT x = 1

The solutions to these equations are $M_{21} = M_{23} = 0$ and (a) = 0. for zero moment at the center support the frequencies of free vibration are equal to the natural frequencies $f_1, f_2, f_3 \ldots\ldots\ldots f_n$ of one member simply supported. The deflection of the continuous beam will have the same shape as the modes of a simply supported beam. For example when the frequency of free vibrations is equal to f_1, the deflection of the left span is $A \sin \frac{\pi x}{l} \sin 2\pi f_1 t$ and the right span $-A \sin \frac{\pi x}{l} \sin 2\pi f_1 t$

For (a) = 0 the frequencies of free vibration are obtained from Fig. 1. This figure shows that (a) = 0 for values of $\frac{f}{f_1}$ = 1.56, 5.07, 10.57, 18.68 .... The fundamental frequency of the member can be computed and therefore the frequencies of free vibration of the two span continuous beam can be obtained from these ratios.

Fig. 3 shows the shape of the deflection curve for $\frac{f}{f_1}$* = 1.56. These deflections are periodic and proportional to the moment at the center. For three identical spans the following equations can be written at supports 2 and 3,

$$(a)\ M_{21} = (a)\ M_{23} + (b)\ M_{32}$$

$$(b)\ M_{23} + (a)\ M_{32} = (a)\ M_{34}$$

$$M_{21} + M_{23} = 0$$

$$M_{32} = M_{34} = 0$$

There are three solutions to these equations,

$$M_{21} = M_{23} = M_{32} = M_{34} = 0$$

$$M_{21} = -M_{23} = M_{32} = -M_{34} \text{ and } \frac{b}{a} = 2$$

$$M_{21} = -M_{23} = -M_{32} = M_{34} \text{ and } \frac{b}{a} = -2$$

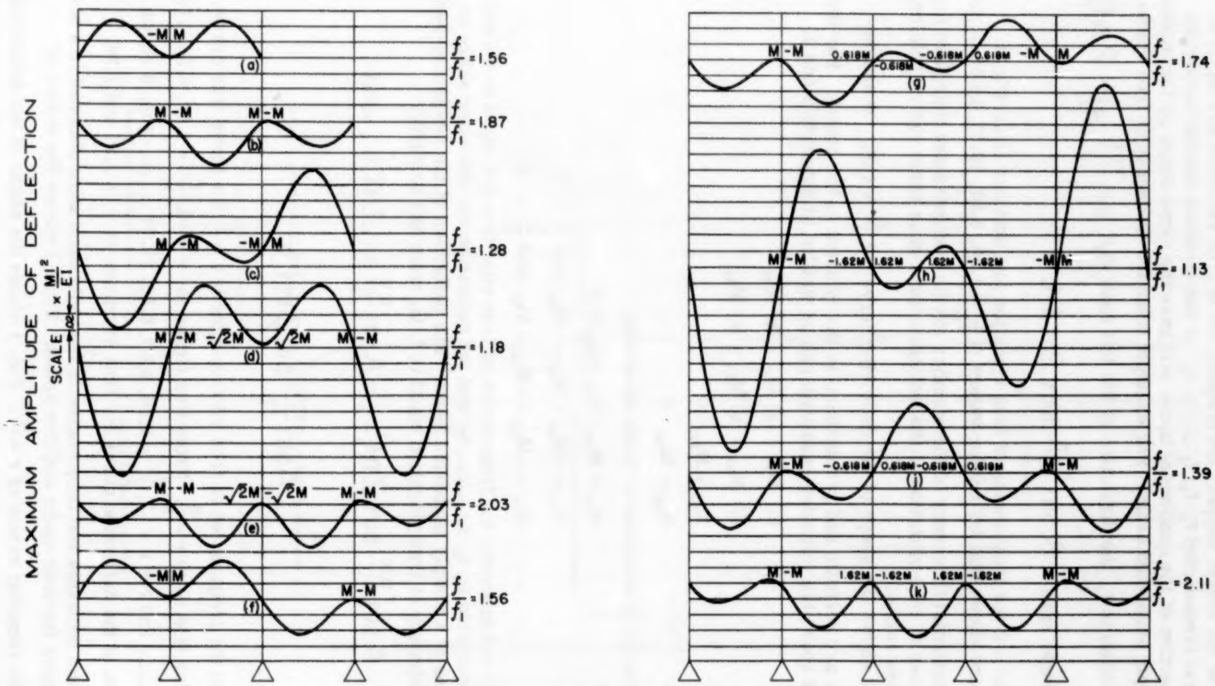
For zero end moments the frequencies of free vibration are equal to the natural frequencies $f_1, f_2, f_3 \ldots\ldots\ldots f_n$ and the deflection has the same shape as the modes of a simply supported beam. The deflection of the continuous beam for a frequency of free vibration equal to f_1 is, from left to right,

$$A \sin \frac{\pi x}{l} \sin 2\pi f_1 t\ ,\ -A \sin \frac{\pi x}{l} \sin 2\pi f_1 t\ , \text{ and}$$

$$A \sin \frac{\pi x}{l} \sin 2\pi f_1 t$$

To obtain the frequencies of free vibration for $b/a = \pm 2$ a curve was drawn, Fig. 4, for this ratio $r = \frac{b}{a}$. From this figure for r = +2, $\frac{f}{f_1}$ = 1.87, 4.58, 11.47, 17.48 .... and for r = -2, $\frac{f}{f_1}$ = 1.28, 5.60, 9.85, 19.77 .... Fig. 3 shows the shape of the deflection curve for free vibration at $\frac{f}{f_1}$ = 1.87 and 1.28.

The end moments and frequencies of free vibration were obtained in a similar manner for four spans and five spans. Table 1 shows the values of $\frac{f}{f_1}$ for the required values of $r = \frac{b}{a}$. Fig. 3 shows the shape of the deflection curve for the lowest frequency of free vibration.



NATURAL MODES FOR IDENTICAL SPANS

FIGURE 3

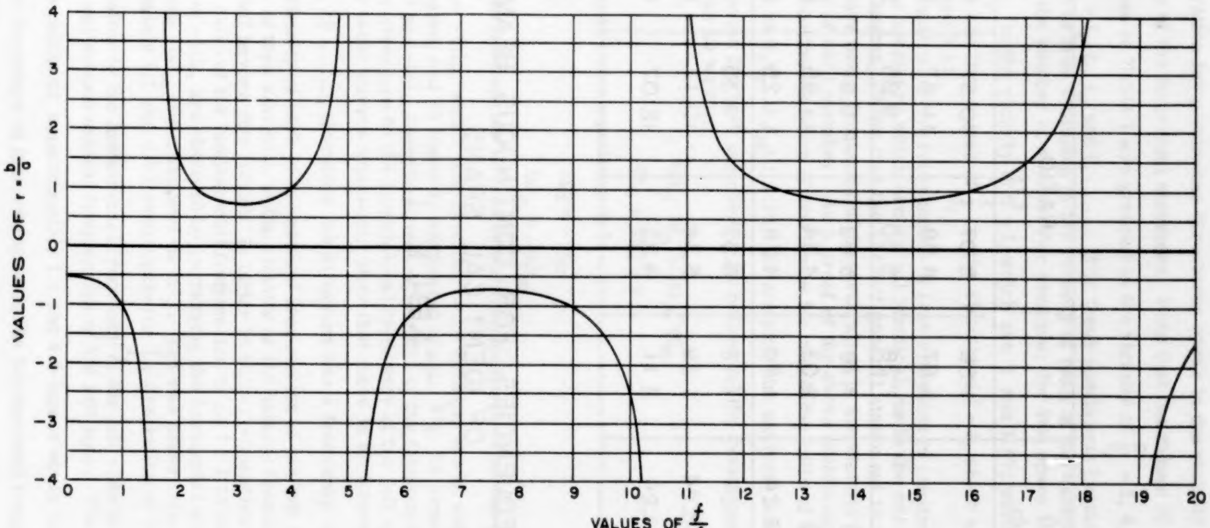


FIGURE 4. THE RATIO OF THE COEFFICIENTS (b) AND (a)

RATIO $r = b/a$	RATIO f/f			
00	1.56	5.07	10.57	18.68
+2	1.87	4.58	11.47	17.48
−2	1.28	5.60	9.85	19.77
$-\sqrt{2}$	1.18	5.88	9.51	
$+\sqrt{2}$	2.03	4.34	11.85	16.92
$1+\sqrt{5}=3.24$	1.74	4.78	11.25	17.93
$1-\sqrt{5}=-1.24$	1.13	5.99	9.35	
$-1-\sqrt{5}=-3.24$	1.39	5.38	10.17	19.36
$-1+\sqrt{5}=1.24$	2.11	4.23	12.07	16.63

FREQUENCIES FOR CONTINUOUS BEAMS
OF IDENTICAL SPANS

TABLE I.

From these studies of free vibration it can be deduced that frequencies of free vibration for continuous structures occur in the vicinity of the natural frequencies of the individual members. Note that the values of $\frac{f}{f_1}$ in the vertical columns of Table 1 are grouped in the vicinity of $\frac{f}{f_1} = 1, 4, 9,$ and 16, ... or $f = f_1, f_2, f_3,$ and f_4 .... For these continuous beams the number frequencies of free vibration in the vicinity of each of the natural frequencies is equal to the number of spans. For example; for two spans there are two values of $\frac{f}{f_1}$ in the vicinity of $f = f_1$ which are 1 and 1.57; while for five spans there are five values of $\frac{f}{f_1}$ in the vicinity of $f = f_1$ which are 1, 1.74, 1.13, 1.39 and 2.11.

An examination of the frequencies of free vibration of continuous structures composed of members which are not all identical, as is the case for these continuous beams, shows that the correct general statement is that a continuous structure has a frequency of free vibration in the vicinity of the natural frequencies of each member. As a proof of the above statement take for example a continuous beam of two spans for which the ratio of the fundamental frequencies is $(f_1)_{12}: (f_1)_{23}: : 1:20$. To simplify the analysis assume l and I to be the same for both members. As before, at the center support, two equations can be written,

$$(a)_{21}\ M_{21} = (a)_{23}\ M_{23}$$

and

$$M_{21} + M_{23} = 0$$

The solution to these equations is [3]

$$M_{21} = -M_{23}$$

and

$$(a)_{21} = -(a)_{23}$$

To obtain the frequencies of free vibration superimpose on the curve for $(a)_{21}$ the curve for $-(a)_{23}$, which will have the same vertical scale, since l, I and M are the same, and 20 times to horizontal scale. Fig. 5A repeats Fig. 1 for $(a)_{21}$ and has $-(a)_{23}$ superimposed. The ratio of the frequency of free vibration of the structure to the fundamental frequency of the left span can be found where these curves intersect. Note that there is an intersection near $\frac{f}{f_1} = 1, 4, 9, 16$ ...., therefore the structure has a frequency of free vibration in the vicinity of each of the natural frequencies of the left span. The frequencies of free vibration in the vicinity of the natural frequencies of the right span fall beyond this figure. In order to show frequencies for both spans assume the ratio of the fundamental frequencies to be 1 : 1.25 instead of 1:20. The values of $-(a)_{23}$ are obtained by increasing the horizontal scale by 1.25. Fig. 5B shows $-(a)_{23}$ superimposed on $(a)_{21}$. The two lower intersection points at approximately 1.1 and 4.3 (lower scale for $\frac{f}{f_1}$) are ratios of the frequency of free vibration to the fundamental frequency of the left span in the vicinity of the first and second natural frequencies of the left span. The two upper intersection points at approximately 1.4 and 4.9 (upper scale for $\frac{f}{f_1}$) are ratios of the frequency of free vibration to the fundamental frequency of the

3. If the square root of the ratio of the fundamental frequencies is a vulgar fraction when $M_{21} = M_{23} = 0$ is a solution.

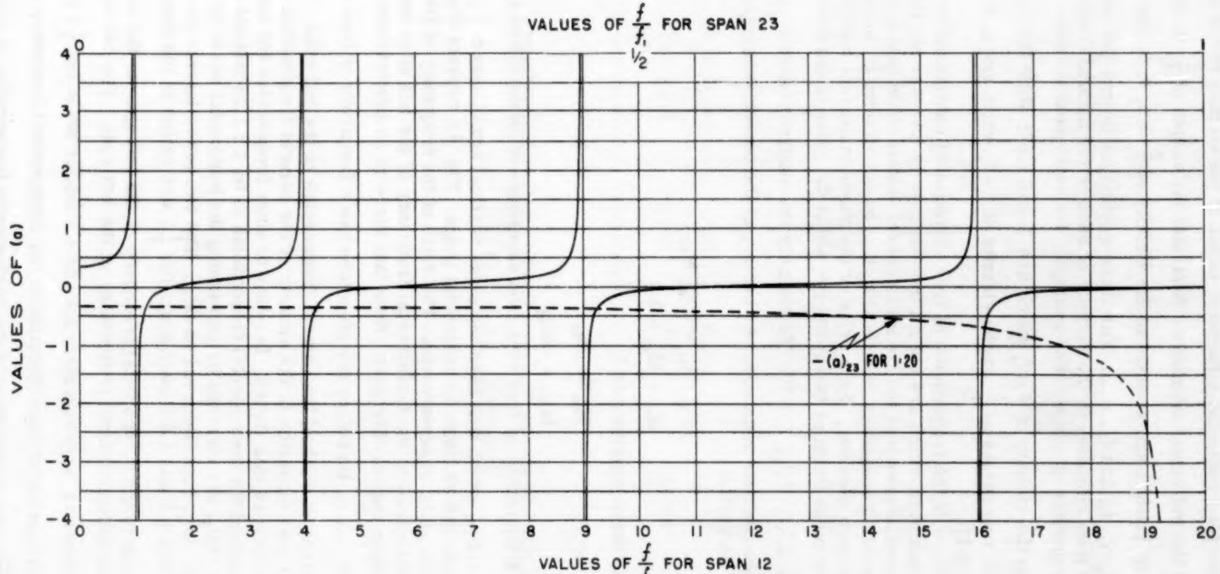


FIGURE 5A. FREQUENCIES FOR TWO SPAN BEAM

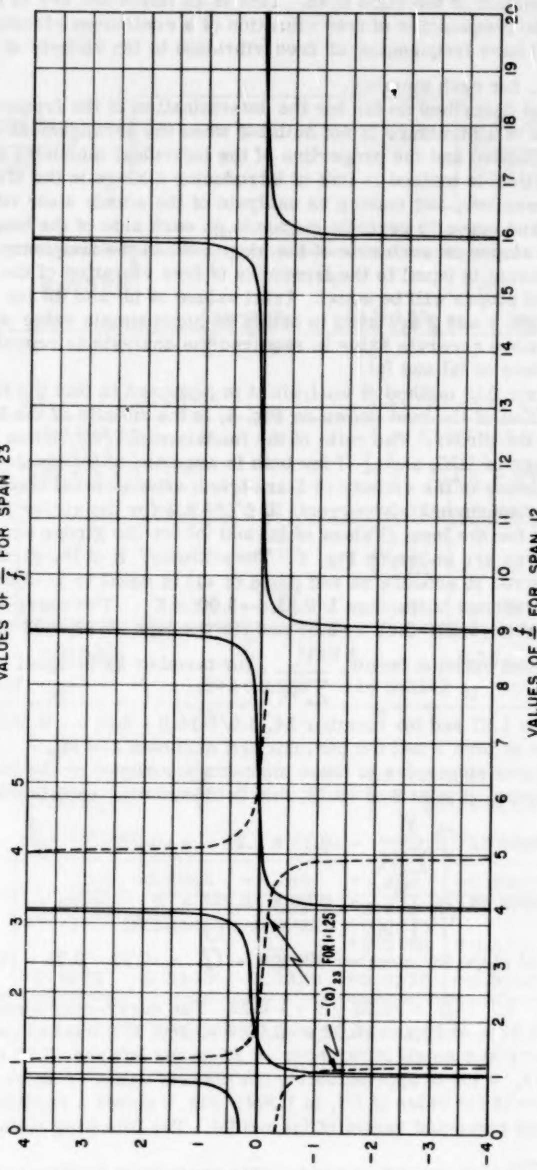


FIGURE 58. FREQUENCIES FOR TWO SPAN BEAM

fundamental frequency of the right span in the vicinity of the first and second natural frequencies of the right span. This is an important key to the determination of the frequencies of free vibration of a continuous structure. The structure will have frequencies of free vibration in the vicinity of $\frac{f}{f_1} = 1$, 4, 9, 16 .... for each member.

The method described so far for the determination of the frequencies of free vibration of a structure is not suitable when the arrangement of the members is complicated and the properties of the individual members differ. A generally applicable method is that of introducing a hinge in the structure at the end of a member, and making an analysis of the steady state vibration due to equal and opposite periodic moments on each side of the hinge to determine the end slopes on each side of the hinge. When the frequency of the periodic moments is equal to the frequency of free vibration of the structure then these end slopes will be equal. Trial values of (a) and (b) for the members from Figs. 1 and 2 are used to obtain an approximate value of the frequency; if a more accurate value is required the analysis is repeated, using computed values of (a) and (b).

To illustrate this method of analysis it is proposed to find the frequency of free vibration of the bent shown on Fig. 6, in the vicinity of the fundamental frequency of the girder. The ratio of the fundamental frequencies of the girder and legs is 1:20; and l/I for both is assumed to be equal. Values of f/f_1 for the girder in the vicinity of 1 are tried; after several trials $f/f_1 = 1.4$ is found to be approximately correct. If $f/f_1 = 1.4$ for the girder then $f/f_1 = 1.4/20 = 0.07$ for the legs. Values of (a) and (b) for the girder and legs taken from the curves are shown on Fig. 6. The stiffness[4] K of the girder (the moment required to produce an end slope of +1) is equal to $1/-0.07 = -14.3 = K_G$ and the stiffness of the legs $1/0.33 = +3.00 = K_L$. The carry-over slope for the girder is $+0.25/-0.07 = -3.57$ and for the legs $-0.17/0.33 = -0.50$. At joint 3, the joint rotation factor, $\frac{K}{\Sigma K}$, for member 32 is equal to $-14.3/(-14.3 + 3.0) = 1.27$ and for member 34, $3.0/(-14.3 + 3.0) = -0.265$. The hinge is introduced at joint 2 and the periodic and moments are $M_{23} = +M$ and $M_{21} = -M$. The end slopes due to these moments are shown on the line ϕ. There is a discontinuity at joint 3 of +0.25, this is distributed as follows

$$\text{Member 32} \quad \phi\frac{K}{\Sigma K} = +0.25 \times 1.27 = +0.32$$

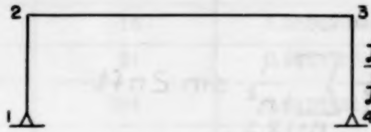
$$\text{Member 34} \quad \phi\frac{K}{\Sigma K} = 0.00 \times -0.265 = \underline{\;0\;}$$

$$\theta = +0.32$$

The change of slope for member 32, $\theta - \phi = +0.32 - 0.25 = +0.07$ and for member 34, $\theta - \phi = +0.32 - 0 = +0.32$. The carry-over slopes are equal to $+0.07 \times -3.57 = -0.25$ and $+0.32 \times -0.50 = -0.16$. The total slopes show that at joint 2 there is a small difference: -0.33 on the left and -0.32 on the right. Therefore $f/f_1 = 1.4$ is approximately the correct value. A more accurate determination of the value of f/f_1 is 1.394; Fig. 6 shows a repetition of the analysis using computed values of (a) and (b). The following method of

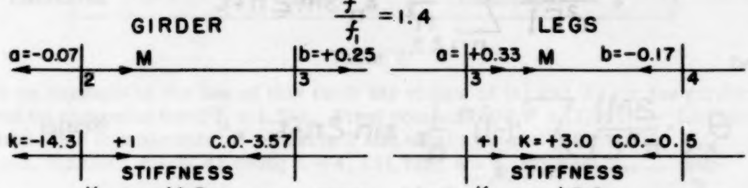
4. The distribution procedure used in this analysis is similar to that described in a paper by L. E. Grinter entitled "Analysis of Continuous Frames by Balancing Angle Changes" Transactions, Am. Soc. C.E., Vol. 107 (1937), p. 1020.

$$\frac{\frac{l}{I}\,(\text{GIRDER})}{\frac{l}{I}\,(\text{LEGS})} = 1$$



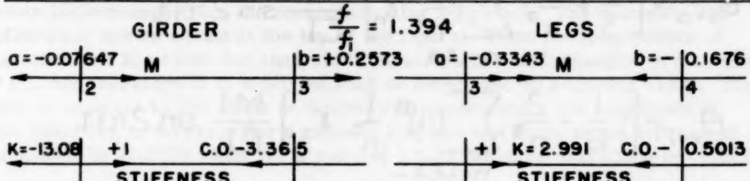
$$\frac{\frac{f}{f_1}\,(\text{GIRDER})}{\frac{f}{f_1}\,(\text{LEGS})} = \frac{1}{20}$$

NO SIDESWAY



$$\frac{K}{\Sigma K} = \frac{-14.3}{-14.3 + 3.0} = 1.27 \qquad \frac{K}{\Sigma K} = \frac{+3.0}{-14.3 + 3.0} = -0.265$$

K/ΣK		-0.265	+1.27	+1.27	-0.265	
C.O.	-0.50		-3.57	-3.57		-0.50
ϕ	+0.17	-0.33	-0.07	+0.25		
$\Theta - \Phi$				+0.07	+0.32	
C.O.			-0.25			-0.16
TOTAL	+0.17	-0.33	-0.32	+0.32	+0.32	-0.16



$$\frac{K}{\Sigma K} = 1.296 \qquad \frac{K}{\Sigma K} = -0.2964$$

K/ΣK		-0.2964	+1.296	+1.296	-0.2964	
C.O.	-0.5013		-3.365	-3.365		-0.5013
Φ	+0.1676	-0.3343	-0.0765	+0.2573		
$\Theta - \Phi$				+0.0762	+0.3335	
C.O.			-0.2564			-0.1672
TOTAL	+0.1676	-0.3343	-0.3329	+0.3335	+0.3335	-0.1672

ANALYSIS FOR FREQUENCY OF BENT

Figure 6

computing (a) and (b) is recommended. The end slopes are obtained by differentiating Equation 1 and substituting $x = 0$ and $x = l$. In addition $\beta = 1 + \gamma_n$ is substituted to separate the series into static and dynamic parts. From Equation 1,

$$\theta_{x=0} = \frac{2Ml}{\pi^2 EI} \sum_{n=1,2,3,\ldots} \frac{1}{n^2} \sin 2\pi ft \qquad \text{(Static)}$$

$$+ \frac{2Ml}{\pi^2 EI} \sum_{n=1,2,3,\ldots} \frac{1}{n^2} \gamma_n \sin 2\pi ft \qquad \text{(Dynamic)}$$

and

$$\theta_{x=l} = \frac{2Ml}{\pi^2 EI} \sum_{n=1,2,3,\ldots} (-1)^n \frac{1}{n^2} \sin 2\pi ft \qquad \text{(Static)}$$

$$+ \frac{2Ml}{\pi^2 EI} \sum_{n=1,2,3,\ldots} (-1)^n \frac{1}{n^2} \gamma_n \sin 2\pi ft \qquad \text{(Dynamic)}$$

Therefore the values of (a) and (b) are

$$\theta_{x=0} = \left[\frac{1}{3} + \frac{2}{\pi^2} \sum_{n=1,2,3,\ldots} \frac{1}{n^2} \gamma_n\right] \frac{Ml}{EI} \sin 2\pi ft$$

$$\theta_{x=l} = \left[-\frac{1}{6} + \frac{2}{\pi^2} \sum_{n=1,2,3,\ldots} (-1)^n \frac{1}{n^2} \gamma_n\right] \frac{Ml}{EI} \sin 2\pi ft$$

Since $\beta_n = 1/(1 - f^2/f_n^2)$ and $f_n^2 = n^2 f_1$, $\frac{2}{\pi^2} \frac{1}{n^2} \gamma_n$ can be written as

$$\frac{2}{\pi^2} \frac{1}{n^2} \frac{1}{n^4\left(\frac{f_1}{f}\right)^2 - 1}$$

and in this form the terms of the series evaluated by means of Table 2 which lists n^4 and $\frac{2}{\pi^2} \frac{1}{n^2}$ for values of n = 1, 2, 3, ...

n	n^4	$\frac{2}{\pi^2} \quad \frac{1}{n^2}$
1	1	0.202642
2	16	0.0506606
3	81	0.0225158
4	256	0.0126651
5	625	0.00710569
6	1296	0.00562895

Table 2

As an example of the use of this table the values of (a) and (b) for the girder will be computed for $f/f_1 = 1.394$. First compute $(f/f_1)^2 = (1.394)^2 = 1.94324$; divide this number into n^4 of Table 2 and obtain for n = 1, 0.514606; n = 2, 8.23369; n = 3, 41.6830; n = 4, 131.739; n = 5, 321.638; .... Subtract one from each of these numbers and multiply the reciprocal by $\frac{2}{\pi^2} \frac{1}{n^2}$ in Table 2 and obtain for n = 1, -0.417480; n = 2, 0.007003; n = 3, 0.000553; n = 4, 0.000097; n = 5, 0.000025; .... Then (a) = 0.333333 - 0.417480 + 0.007003 + 0.000553 + 0.000097 + 0.000025 = -0.076469; and (b) = -0.166667 + 0.417480 + 0.007003 - 0.000553 + 0.000097 - 0.000025 = +0.257335.

The computation on Fig. 6 for $f/f_1 = 1.394$ using these values of (a) and (b) shows little change from the computation for $f/f_1 = 1.4$. The moments and deflections are as shown at the top of the figure. When the frequencies of the members are close the analysis is sensitive to small changes in the value of f/f_1 and therefore it is more difficult to determine an accurate value. The ratio of moment to end slope is larger, and consequently the amplitude of free vibration is larger. For a ratio of 1:20 the end slope at the left end of the girder is -0.3329 while for a ratio of 1:1.25 the same end slope is -1.301.

3. Influence Lines.

The behavior of structures, in particular the influence of the position of the periodic force, can be studied most conveniently by means of influence lines. The proof of the validity of the use of superposition in the analysis for steady state forced vibration is inherent in the proof of Harmonic Analysis and will not be repeated here. Both the reciprocal theorem and Muller-Breslau's principle can be applied to the study of vibrations. It is necessary to supplement the usual concepts because of the additional factors in the analysis of time and damping. The reciprocal theorem is valid if the periodic forces have the same frequency and are in phase. The deflection at any point of a simply supported beam due to a periodic force P sin $2\pi ft$ at a distance, s, from the left end is given by the following equation,

$$y = \frac{2Pl^3}{\pi^4 EI} \sum_{n=1,2,3,\ldots} \frac{1}{n^4} \beta_n \sin\frac{n\pi s}{l} \sin\frac{n\pi x}{l} \sin(2\pi ft - \alpha_n)$$

where,

$$\beta_n = \frac{1}{\sqrt{\left[1-\left(\frac{f}{f_n}\right)^2\right]^2 + \left(2\frac{\delta}{f_n}\frac{f}{f_n}\right)^2}}$$

$$\tan\alpha_n = \frac{2\frac{\delta}{f_n}\frac{f}{f_n}}{1-\frac{f^2}{f_n^2}}$$

The deflection at (a) due to a periodic force at (b) is equal to,

$$Y_{ab} = \frac{2P_b l^3}{\pi^4 EI} \sum_{n=1,2,3,\dots} \frac{1}{n^4}\beta_n \sin\frac{n\pi s_b}{l}\sin\frac{n\pi s_a}{l}\sin(2\pi ft - \alpha_n) ;$$

this deflection and the phase angle are shown on Fig. 7 by means of a vector diagram. The deflection at (b) due to a periodic force at (a) is,

$$Y_{ba} = \frac{2P_a l^3}{\pi^4 EI} \sum_{n=1,2,3,\dots} \frac{1}{n^4}\beta_n \sin\frac{n\pi s_a}{l}\sin\frac{n\pi s_b}{l}\sin(2\pi ft - \alpha_n) ;$$

this deflection and phase angle are shown on Fig. 7. Therefore $Y_{ab} = Y_{ba}$ and these deflections, which are periodic, are always equal, and in phase if the periodic forces are equal and in phase. The reciprocal theorem is also valid for rotations and deflections. The rotation at (a), shown on Fig. 7, due to the periodic force at (b) is obtained by differentiating equation 3 and substituting $x = 0$,

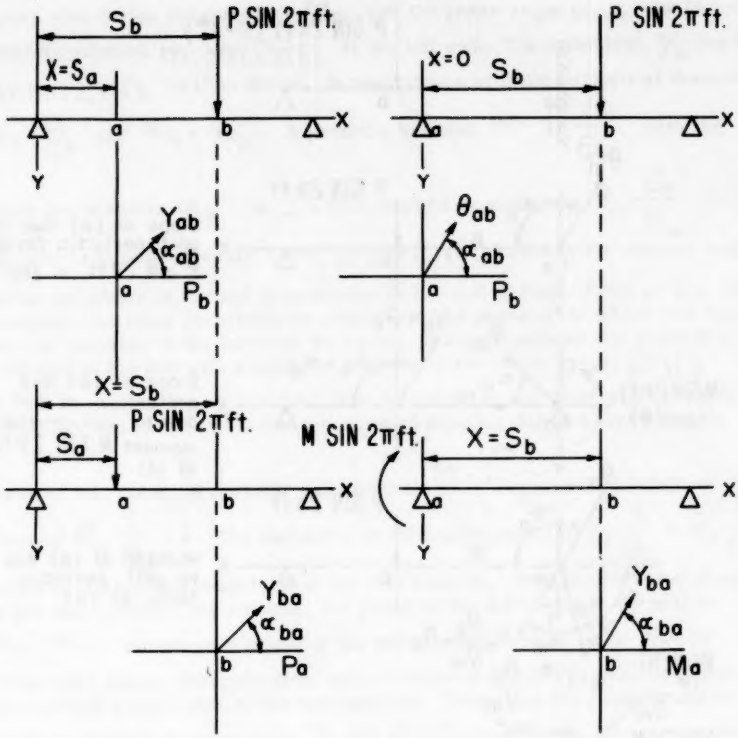
$$\theta_{ab} = \left(\frac{dy}{dx}\right)_{x=0} = \frac{2P_b l^2}{\pi^3 EI} \sum_{n=1,2,3,\dots} \frac{1}{n^3}\beta_n \sin\frac{n\pi s_b}{l}\sin(2\pi ft - \alpha_n)$$

The deflection at (b) due to a periodic moment at (a) is given by the equation,

$$Y_{ba} = \frac{2M_a l^2}{\pi^3 EI} \sum_{n=1,2,3,\dots} \frac{1}{n^3}\beta \sin\frac{n\pi s_b}{l}\sin(2\pi ft - \alpha_n)$$

which is the general equation with S_b substituted for x. Therefore $\theta_{ab} = Y_{ba}$ in magnitude and phase at all times when the periodic force $P_b \sin 2\pi ft$ and the periodic moment $M_a \sin 2\pi ft$ are equal and in phase.

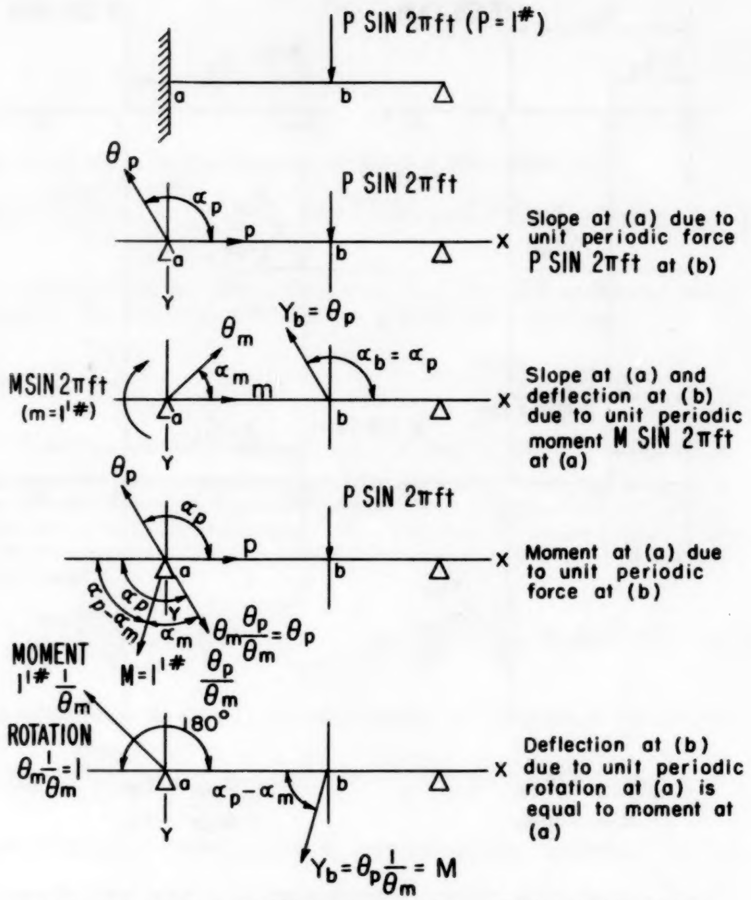
Muller-Breslau's principle will be proven in the usual manner, using the reciprocal theorem. Fig. 8 shows a beam fixed at the left end, and simply supported at the right. The moment at the left end due to a unit periodic force at any point (b) will be determined; it will then be proven that this moment is equal to the deflection at (b) due to a unit periodic end rotation. The moment at the end due to the unit periodic force is found by making the left end simply supported and finding the end slope due to the periodic force,



$Y_{ab} = Y_{ba}$
$\alpha_{ab} = \alpha_{ba}$

$\theta_{ab} = Y_{ba}$
$\alpha_{ab} = \alpha_{ba}$

RECIPROCAL THEOREM
FIGURE 7



MULLER - BRESLAU'S PRINCIPLE
FIGURE 8

then applying a periodic moment at the end sufficient to make the end slope zero.

The end slope θ_p and the phase angle α_p in relation to the force $p \sin 2\pi ft$ are shown by means of the rotating vector diagram. This figure also shows the end slope θ_m and the phase angle α_m due to a unit periodic moment $m \sin 2\pi ft$ at the left end. The deflection Y_b due to $m \sin 2\pi ft$ is also shown. In accordance with the reciprocal theorem, $Y_b = \theta_p$ and $\alpha_b = \alpha_p$. A periodic moment $M = 1^{*}\frac{\theta_p}{\theta_m}$ with the phase angle shown, $\alpha_p - \alpha_m$, will produce an end slope $\theta_m \frac{\theta_p}{\theta_m} = \theta_p$

θ_p due to the unit periodic force $p \sin 2\pi ft$ This is the correct magnitude and phase of the end moment due to the unit periodic force at (b).. The influence line must show both the magnitude and phase of M which will vary with the position of the force on the beam. If a unit periodic end rotation is produced at the end with a phase in relation to the force $p \sin 2\pi ft$ of 180° (opposite to p as a vector), the deflection at any point will be equal in magnitude and phase at all times to the end moment due to a unit periodic force at that point. A periodic end moment $M = 1^{*}\frac{1}{\theta_m}$ will produce an end slope of $\theta_m \frac{1}{\theta_m} = 1$. The deflection at (b) will be equal to $Y_b \frac{1}{\theta_m} = \theta_p \frac{1}{\theta_m}$ which is the correct magnitude of the end moment. With the required phase of the unit periodic end rotation, the phase of the deflection at (b) will be $\alpha_p - \alpha_m$, the correct phase of the end moment. This can be done by means of a double influence line: the ordinates to one curve will represent the vertical components to the end moment. These are the components relative to the unit periodic force $p \sin 2\pi ft$ when t = 0. The maximum value of the end moment at any point will be equal to the square root of the sum of the squares of the two component and the tangent of the phase angle of the end moment relative to the unit periodic force will be equal to the vertical ordinate divided by the horizontal ordinate.

The computation of the influence line ordinates will be described in detail for this beam. The procedure recommended is as follows,

1. Compute the horizontal and vertical components of the end slope due to a moment $M \sin 2\pi ft$.
2. Compute the horizontal and vertical components of the deflection at equally spaced interval due to a moment $M \sin 2\pi ft$.
3. From (1) above find the horizontal and vertical components of the end moment required to produce a unit periodic end rotation.
4. From (2) and (3) above find the ordinates to the influence lines.

This procedure with some minor modification is generally applicable to continuous structures.

The deflection due to a periodic moment $M \sin 2\pi f t$ is given by Equation 4. This equation is separated into horizontal and vertical components by substituting

$$\sin(2\pi f t - \alpha_n) = \cos\alpha_n \sin 2\pi f t - \sin\alpha_n \cos 2\pi f t$$

$$y = \frac{2Ml^2}{\pi^3 EI} \sum_{n=1,2,3,\ldots} \frac{1}{n^3} \beta_n^2 \left(1 - \frac{f^2}{f_n^2}\right) \sin\frac{n\pi x}{l} \sin 2\pi f t$$

$$- \frac{2Ml^2}{\pi^3 EI} \sum_{n=1,2,3,\ldots} \frac{1}{n^3} \beta_n^2 \left(2\frac{\delta}{f_n}\frac{f}{f_n}\right) \sin\frac{n\pi x}{l} \cos 2\pi f t$$

For convenience in computation substitute $\beta_n^2\left(1 - \frac{f^2}{f_n^2}\right) = 1 + \gamma_n'$:

$$y = \frac{2Ml^2}{\pi^3 EI} \sum_{n=1,2,3,\ldots} \frac{1}{n^3} \sin\frac{n\pi x}{l} \sin 2\pi f t \qquad \text{(Static)}$$

$$+ \frac{2Ml^2}{\pi^3 EI} \sum_{n=1,2,3,\ldots} \frac{1}{n^3} \gamma_n' \sin\frac{n\pi x}{l} \sin 2\pi f t \qquad \text{(Dynamic)}$$

$$- \frac{2Ml^2}{\pi^3 EI} \sum_{n=1,2,3,\ldots} \frac{1}{n^3} \beta_n^2 \left(2\frac{\delta}{f_n}\frac{f}{f_n}\right) \sin\frac{n\pi x}{l} \cos 2\pi f t \,. \qquad \text{(Dynamic)}$$

The horizontal and vertical components of the end slope are obtained by differentiating this equation for the deflection and substituting x = o:

$$\theta_{x=0} = \left[\frac{1}{3} + \frac{2}{\pi^2} \sum_{n=1,2,3,\ldots} \frac{1}{n^2} \gamma_n'\right] \frac{Ml}{EI} \sin 2\pi f t \qquad \text{(Horizontal)}$$

$$- \left[\frac{2}{n^2} \sum_{n=1,2,3,\ldots} \frac{1}{n^2} \beta_n^2 \left(2\frac{\delta}{f_n}\frac{f}{f_n}\right)\right] \frac{Ml}{EI} \cos 2\pi f t \qquad \text{(Vertical)}$$

The horizontal and vertical components of the deflection are:

$$y = [\text{STATIC DEFLECTION}]\frac{Ml^2}{EI}\sin 2\pi ft$$

$$\left[\frac{2}{\pi^3}\sum_{n=1,2,3,\dots}\frac{1}{n^3}\gamma_n'\sin\frac{n\pi x}{l}\right]\frac{Ml^2}{EI}\sin 2\pi ft$$

$$-\left[\frac{2}{\pi^3}\sum_{n=1,2,3,\dots}\frac{1}{n^3}\beta_n^2\, 2\frac{\delta}{f_n}\frac{f}{f_n}\sin\frac{n\pi x}{l}\right]\frac{Ml^2}{EI}\cos 2\pi ft.$$

The horizontal components of the deflection are equal to the static deflection due to a moment on the end plus the ordinates tc the curves

$$\frac{2}{\pi^3}\frac{1}{3^3}\gamma_3'\sin\frac{3\pi x}{l}$$

$$\frac{2}{\pi^3}\frac{1}{1^3}\gamma_1'\sin\frac{\pi x}{l},\ \frac{2}{\pi^3}\frac{1}{2^3}\gamma_2'\sin\frac{2\pi x}{l},$$

These components will be designated as Curve S and are multiplied by the $\sin 2\pi ft$. **The vertical components of the deflection are equal to the sum of the curves**

$$-\frac{2}{\pi^3}\frac{1}{1^3}\beta_1^2\left(2\frac{\delta}{f_1}\frac{f}{f_1}\right)\sin\frac{\pi x}{l},\ -\frac{2}{\pi^3}\frac{1}{2^3}\beta_2^2\left(2\frac{\delta}{f_1}\frac{f}{f_1}\right)\sin\frac{2\pi x}{l}$$

$$-\frac{2}{\pi^3}\frac{1}{3^3}\beta_3^2\, 2\frac{\delta}{f_1}\frac{f}{f_1}\sin\frac{3\pi x}{l}$$

These components will be designated as Curve C and are multiplied by the $\cos 2\pi ft$.

The influence line for the moment at the end will be computed for a beam with a natural frequency $f_1 = 4$ and a damping coefficient $\delta = 0.08$ **The unit periodic force** $p\sin 2\pi ft$ **will be assumed to have a frequency $f = 4$. It is necessary to compute** γ_n' **and** $\beta_n^2\ 2\frac{\delta}{f_1}\frac{f}{f_1}$ **the coefficients which appear in the series, for values of n = 1, 2, 3 .... First** β_n^2 **which is equal to the reciprocal of** $\left[1-\left(\frac{f}{f_n}\right)^2\right]^2 + \left(2\frac{\delta}{f_n}\frac{f}{f_n}\right)^2$ **is computed for n = 1, 2, 3 and 4. This computation is shown on Table 3 and the**

LINE	TERM	$n=1$	$n=2$	$n=3$	$n=4$
1	$\frac{f}{f_n}$	4/4	4/16	4/36	4/64
2	$\left\lvert\frac{f}{f_n}\right\rvert^2$	1.000000	0.0625000	0.0123457	0.00390625
3	$1-\left\lvert\frac{f}{f_n}\right\rvert^2$	0.000000	0.9375000	0.987654	0.996094
4	$\left[1-\left\lvert\frac{f}{f_n}\right\rvert^2\right]^2$	0.000000	0.878906	0.975461	0.992203
5	$2\frac{\delta}{f_n}$	0.04	0.01	0.004444	0.0025
6	$2\frac{\delta}{f_n}\frac{f}{f_n}$	0.04	0.0025	$0.\overset{3}{0}493827$	$0.\overset{3}{0}156250$
7	$\left\lvert 2\frac{\delta}{f_n}\frac{f}{f_n}\right\rvert^2$	0.0016	$0.\overset{5}{0}625$	$0.\overset{6}{0}24$	$0.\overset{7}{0}24$
8	$\left[1-\left\lvert\frac{f}{f_n}\right\rvert^2\right]^2+\left\lvert 2\frac{\delta}{f_n}\frac{f}{f_n}\right\rvert^2$	0.0016	0.878912	0.975461	0.992203
9	β_n^2	625	1.137770	1.0251561	1.00785837
10	$\beta_n^2\left[1-\left\lvert\frac{f}{f_n}\right\rvert^2\right]$	0.000000	1.0666592	1.0124998	1.00392147
11	γ_n^1	−1	0.0666592	0.0124998	0.00392147
12	$\beta_n^2\left\lvert 2\frac{\delta}{f_n}\frac{f}{f_n}\right\rvert$	25	$0.\overset{2}{0}28442$	$0.\overset{3}{0}506250$	$0.\overset{3}{0}157478$
13	$\frac{2}{\pi^3}\frac{1}{n^3}\gamma_n^1$	−0.0645031	$0.\overset{3}{0}537465$	$0.\overset{4}{0}298621$	$0.\overset{5}{0}395232$
14	$-\frac{2}{\pi^3}\frac{1}{n^3}\beta_n^2\left\lvert 2\frac{\delta}{f_n}\frac{f}{f_n}\right\rvert$	−1.612577	$-0.\overset{4}{0}229343$	$-0.\overset{5}{0}120943$	
15	$\frac{2}{\pi^2}\frac{1}{n^2}\gamma_n^1$	−0.202642	$0.\overset{2}{0}337700$	$0.\overset{3}{0}281443$	
16	$-\frac{2}{\pi^2}\frac{1}{n^2}\beta_n^2\left\lvert 2\frac{\delta}{f_n}\frac{f}{f_n}\right\rvert$	−5.06605	$-0.\overset{3}{0}144100$	$-0.\overset{4}{0}113986$	

COMPUTATION OF γ_n^1 AND $\beta_n^2 \; 2\frac{\delta}{f}\frac{f}{f}$ FOR $f_n = 4$, $f = 4$, $\delta = 0.08$

TABLE 3.

the values of β_n^2 are on line 9. The values of $\beta_n^2\left(1-\frac{f^2}{f_n^2}\right)$ are next computed by multiplying line 9 by line 3. γ_n^1 shown on line 11 is obtained by subtracting 1 from line 10. $\beta_n^2\left(2\,\frac{\delta}{f_n}\,\frac{f}{f_n}\right)$ shown on line 12 is obtained by multiplying line 9 by line 6. The values of the terms in the series are shown on lines 13, 14, 15 and 16.

The end slope due to a moment $M \sin 2\pi ft$ is equal to 0.333333 - 0.202642 + 0.003377 + 0.000281[1] = +0.134349 $\frac{Ml}{EI}\sin 2\pi ft$ for the horizontal component, and to -5.06605 - 0.00014 - 0.00001[2] = -5.06621 $\frac{Ml}{EI}\cos 2\pi ft$ for the vertical component.

The horizontal and vertical component of the deflection due to a moment $M \sin 2\pi ft$ are shown computed on Table 4. The horizontal components due to a clockwise moment on the left end are equal to the static deflection plus -0.0645031 $\sin\frac{\pi x}{l}$, , 0.000537465 $\sin\frac{2\pi x}{l}$ and 0.0000298621 $\sin\frac{3\pi x}{l}$[3]. The sum of these ordinates is called Curve S and the deflection is equal to (Curve S) $\frac{Ml^2}{EI}\sin 2\pi ft$ The vertical components are equal to -1.61258 $\sin\frac{\pi x}{l}$ and -0.0000229343 $\sin\frac{2\pi x}{l}$[4]........ The sum of these ordinates is called Curve C and the deflection is equal to (Curve C) $\frac{Ml^2}{EI}\cos 2\pi ft$.

The horizontal component, M_H, and vertical component M_V, of the end moment required to produce a unit periodic end rotation are obtained by means of the following equations.

$$0.134349\ M_H + 5.06620\ M_V = -1$$
$$-5.06620\ M_H + 0.134349\ M_V = 0$$

1. See Table 3, line 15
2. See Table 3, line 16
3. See Table 3, line 13
4. See Table 3, line 14

$\frac{x}{l}$	STATIC	-0.0645031 $x \sin \frac{\pi x}{l}$	$\overset{3}{0}.0537465$ $x \sin \frac{2\pi x}{l}$	$\overset{4}{0}.0298621$ $x \sin \frac{3\pi x}{l}$	ORDINATES CURVE S	-1.61258 $x \sin \frac{\pi x}{l}$	$-\overset{4}{0}.02293$ $x \sin \frac{2\pi x}{l}$	$-\overset{5}{0}.01209$ $x \sin \frac{3\pi x}{l}$	ORDINATES CURVE C
0/12	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.00000	0.00000	0.00000	0.00000
1/12	0.0244020	-0.0166946	0.0002687	0.0000211	0.0079972	-0.41737	-0.00001	0.00000	-0.41738
2/12	0.0424383	-0.0322515	0.0004655	0.0000299	0.0106822	-0.80629	-0.00002	0.00000	-0.80631
3/12	0.0546875	-0.0456106	0.0005375	0.0000211	0.0096355	-1.14026	-0.00002	0.00000	-1.14028
4/12	0.0617284	-0.0558613	0.0004655	0.0000000	0.0063326	-1.39653	-0.00002	0.00000	-1.39655
5/12	0.0614397	-0.0623052	0.0002687	-0.0000211	-0.0006179	-1.55763	-0.00001	0.00000	-0.55764
6/12	0.0625000	-0.0645031	0.0000000	-0.0000299	-0.0020330	-1.61258	0.00000	0.00000	-1.61258
7/12	0.0573881	-0.0623052	-0.0002687	-0.0000211	-0.0052069	-1.55763	0.00001	0.00000	-1.55762
8/12	0.0493827	-0.0558613	-0.0004655	0.0000000	-0.0069441	-1.39653	0.00002	0.00000	-1.39651
9/12	0.0390625	-0.0456106	-0.0005375	0.0000211	-0.0070645	-1.14026	0.00002	0.00000	-1.14024
10/12	0.0270062	-0.0322515	-0.0004655	0.0000299	-0.0056809	-0.80629	0.00002	0.00000	-0.80627
11/12	0.0137924	-0.0166946	-0.0002687	0.0000211	-0.0031498	-0.41737	0.00001	0.00000	-0.41736
12/12	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.00000	0.00000	0.00000	0.00000

HORIZONTAL AND VERTICAL COMPONENTS OF DEFLECTION DUE TO $M \sin 2\pi ft$ at $t = 0$

TABLE 4

Solving these two equations, the horizontal component $M_H = -0.00523267$ $\frac{EI}{l}\sin 2\pi ft$ and the vertical component $M_V = -0.197247$

$\frac{EI}{l}\cos 2\pi ft$ In order to find the horizontal and vertical component of the influence line ordinates the ordinates to Curve S and Curve C, which are for end moment $M \sin 2\pi ft$, are multiplied by the moments above in accordance with the following table.

Deflection Ordinates	End Moment			
	$M \sin 2\pi ft$	$-M \sin 2\pi ft$	$M \cos 2\pi ft$	$-M \cos 2\pi ft$
$[\text{Curve S}]\frac{Ml^2}{EI}$	$x \sin 2\pi ft$	$x - \sin 2\pi ft$	$x \cos 2\pi ft$	$x - \cos 2\pi ft$
$[\text{Curve C}]\frac{Ml^2}{EI}$	$x \cos 2\pi ft$	$x - \cos 2\pi ft$	$x - \sin 2\pi ft$	$x \sin 2\pi ft$

The above end moments cause the following deflections, due to $M_H = -.00523267$ $\frac{EI}{l}\sin 2\pi ft$ the deflection is (Curve S) (0.00523267) $l(-\sin 2\pi ft)$ and (Curve) (0.00523267) $l(-\cos 2\pi ft)$; due to $M_V = -0.197247$ $\frac{EI}{l}\cos 2\pi ft$ and deflection is (Curve S) (0.197247) $l(-\cos 2\pi ft)$ and (Curve C) (0.197247) $l(\sin 2\pi ft)$. The sine terms above are the influence ordinates for the horizontal component of the end moment and the cosine terms above are the influence ordinates for the vertical component of the end moment. The computation of the above ordinates is shown in Table 5, and plotted on Fig. 9. The end moment due to a periodic force $P \sin 2\pi ft$ at the center is shown with the phase angle relative to the force at $t = 0$.

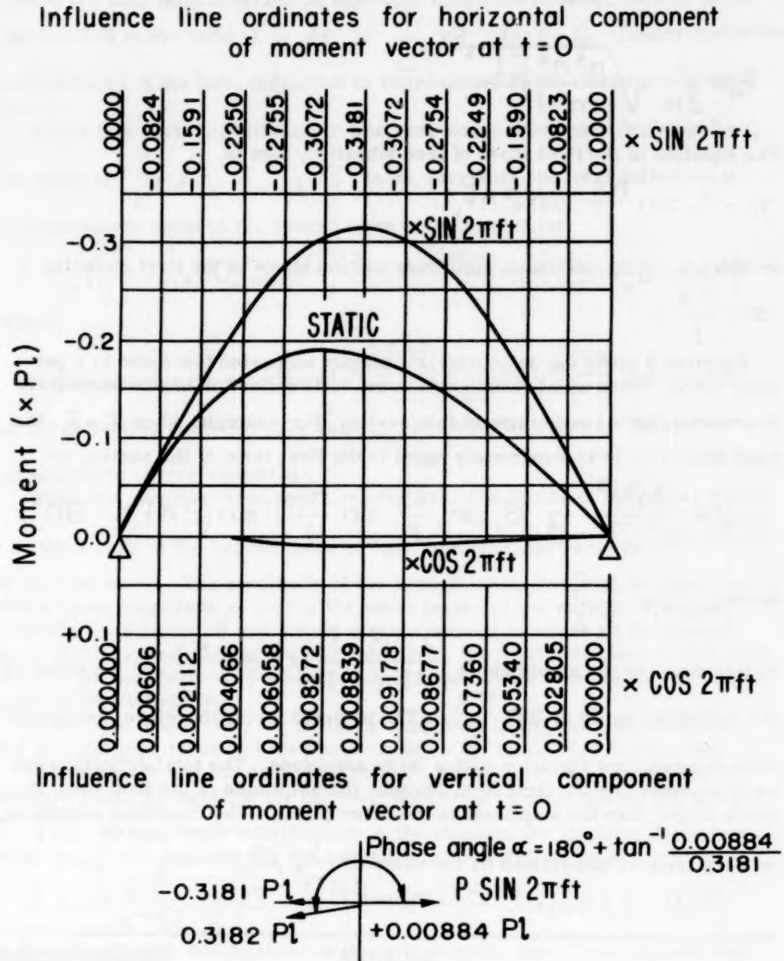
4. Behavior of Continuous Structures

The behavior of continuous structures when synchronism exists is of primary importance. When synchronism exists between the frequency of the periodic force and the natural frequency of the structure, the force will excite a mode of free vibration with a large amplitude. This behavior can be described in detail for a simply supported beam by means of Equation 3. The behavior of a continuous structure is similar to that of a simply supported beam. This similarity in behavior will be shown, using as an example a three-span continuous beam.

$\frac{x}{l}$	$M_v = -0.197247 \frac{EI}{l} \cos 2\pi ft$		$M_h = -0.00523267 \frac{EI}{l} \sin 2\pi ft$		INFLUENCE LINE FOR HORIZONTAL COMPONENT	INFLUENCE LINE FOR VERTICAL COMPONENT
	(CURVE S) $(0.197247)\, l$ $(-\cos 2\pi ft)$	(CURVE C) $(0.197247)\, l$ $(\sin 2\pi ft)$	(CURVE S) $(0.0053267)\, l$ $(-\sin 2\pi ft)$	(CURVE C) $(0.00523267)\, l$ $(-\cos 2\pi ft)$	$(\sin 2\pi ft)$	$(\cos 2\pi ft)$
0/12	0.00000000	0.000000	0.000000	0.00000000	0.000000	0.00000000
1/12	−0.00157743	−0.082325	−0.000042	0.00218396	−0.082367	0.00060653
2/12	−0.00210703	−0.159038	−0.000056	0.00421905	−0.159094	0.00211202
3/12	−0.00190058	−0.224193	−0.000050	0.00596660	−0.224963	0.00406602
4/12	−0.00124909	−0.275462	−0.000033	0.00730758	−0.275495	0.00605849
5/12	0.00012188	−0.307238	0.000003	0.00815056	−0.307235	0.00827244
6/12	0.00040100	−0.318077	0.000011	0.00843810	−0.318066	0.00883910
7/12	0.00102705	−0.307238	0.000027	0.00815056	−0.307211	0.00917761
8/12	0.00136970	−0.275462	0.000036	0.00730758	−0.275426	0.00867728
9/12	0.00139345	−0.224193	0.000037	0.00596660	−0.224876	0.00736005
10/12	0.00112054	−0.159038	0.000030	0.00421905	−0.159008	0.00533959
11/12	0.00062129	−0.082325	0.000016	0.00218396	−0.082309	0.00280525
12/12	0.0000000	0.000000	0.000000	0.00000000	0.000000	0.00000000

INFLUENCE LINE ORDINATES FOR HORIZONTAL AND VERTICAL COMPONENTS OF END MOMENT AT $t = 0$

TABLE 5



<u>End moment due to force P SIN 2πft at center</u>

Influence line for periodic forced vibration

FIGURE 9

The equation for the free vibration of a simply supported beam is,

$$y_n = A_n \sin \frac{n\pi x}{l} \sin 2\pi f_n t \qquad \text{(EQUATION 5)}$$

$$f_n = \frac{1}{2\pi} \sqrt{\frac{n^4 \pi^4 EI}{m\, l^4}}$$

The equation of the first mode of free vibration, then is,

$$y_1 = A_1 \sin \frac{\pi x}{l} \sin 2\pi f_1 t$$

in which A_1 is the maximum amplitude and the shape of the first mode is $\sin \frac{\pi x}{l}$.

Equation 3 gives the deflection of a simply supported beam due to a periodic force. When synchronism exists the total deflection can be accurately represented by a single term of this series. For example, when $f = f_1$ the total deflection is approximately equal to the first term of the series, or

$$y \cong \frac{2Pl^3}{\pi^4 EI} \frac{1}{1^4} \beta_1 \sin \frac{\pi s}{l} \sin \frac{\pi x}{l} \sin(2\pi f_1 t - 90°)$$

where, $\beta_1 = \frac{1}{2} \frac{f_1}{\delta}$.

In this equation the amplitude is $\frac{2Pl^3}{\pi^4 EI} \frac{1}{1^4} \beta_1 \sin \frac{\pi s}{2}$ and the shape of the deflection curve is $\sin \frac{\pi x}{2}$. The periodic force, therefore, excites the first mode of free vibration with a large amplitude. The total deflection can be represented by the first term because the amplitude of the first term is much larger than the amplitude of the other terms. The maximum amplitude of each term is determined by the value of $\frac{1}{n^4} \beta_n$

Table 6

Frequency	Deflection $\frac{1}{n^4} \beta_n$	Bending Moment $\frac{1}{n^2} \beta_n$
$f = f_1$	375	94
$f = f_2$	94	181
$f = f_3$	180	274
$f = f_4$	273	369

$$\text{Ratio } \frac{\text{Largest term}}{\text{Next largest term}} \text{ FOR } f_1 = 4 \text{ AND } \delta = 0.08$$

Table 6 shows the ratio of the maximum amplitude of the largest term to that of the next largest term. In this table under the heading, deflection the number 375 is the ratio of $\frac{1}{1^4}\beta_1$ to $\frac{1}{2^4}\beta_2$ for $f = f_1$. Clearly the error will be small if the total deflection is represented by the first term of the series.

There is a certain limitation to this reasoning. For example, when $f = f_2$ the ratio of $\frac{1}{2^4}\beta_2$ to $\frac{1}{1^4}\beta$, is 94, therefore the total deflection is approximately equal to the second term of the series, or

$$y \cong \frac{2Pl^3}{\pi^4 EI} \frac{1}{2^4}\beta_2 \sin\frac{2\pi s}{l} \sin\frac{2\pi x}{l} \sin(2\pi f_2 t - 90°)$$

where, $\beta_2 = \frac{1}{2}\frac{f_2}{\delta}$

In this equation the amplitude is $\frac{2Pl^3}{\pi^4 EI}\frac{1}{2^4}\sin\frac{2\pi s}{l}$ and the shape of the deflection curve is $\sin\frac{2\pi x}{l}$ which is the same as that of the second mode of free vibration.

When the periodic force is at, or very near the center of the beam, where a mode occurs in the second mode of free vibration, the value of $\sin\frac{2\pi s}{l}$ is zero or small. The amplitude of the second term, then, can be small and of the same magnitude as that of the other terms of the series. The total deflection will be small and cannot be represented as above by the second term of the series. Subject to this limitation, the total deflection at synchronism can be represented by a single term of the series with the shape of the mode of free vibration.

It has been shown that, for a simply supported beam, the total deflection due to a periodic force has the same shape as the mode of free vibration.

The influence line for deflection at any point x^1 from the left end can be obtained from the equation for the total deflection by substituting $x = x'$ and $P = 1\,lb$. Making these substitutions in the equation for the total deflection when $f = f_1$ the equation for the influence line for deflection at $x = x^1$ is,

$$I.L. \cong \frac{2l^3}{\pi^4 EI}\frac{1}{1^4}\beta_1 \sin\frac{\pi s}{l} \sin\frac{\pi x'}{l} \sin(2\pi f_1 t - 90°)$$

In this equation, s the distance of the unit periodic force from the left end is a variable and the shape of the influence line is $\sin\frac{\pi s}{l}$. Therefore the influence line for deflection has approximately the same shape as the mode of free vibration.

The bending moment at any point, derived from the equation for the total deflection by use of the relation $EI\frac{d^2y}{dx^2} = -M$ is,

$$M \cong \frac{2Pl}{\pi^2} \frac{1}{l^2} \beta_1 \sin\frac{\pi s}{l} \sin\frac{\pi x}{l} \sin(2\pi f_1 t - 90°)$$

The influence line for bending moment at any point a distance x^1 from the left end is obtained by substituting in the above equation, $x = x^1$, and $P = 1$ lb.

which obtains $I.L. \cong \frac{2l}{\pi^2} \frac{1}{l^2} \beta_1 \sin\frac{\pi s}{l} \sin\frac{\pi x'}{l} \sin(2\pi f_1 t - 90°)$

In this equation, s, the distance of the unit periodic force from the left end is a variable and the shape of the influence line is $\sin\frac{\pi s}{l}$. Therefore the influence line for bending moment has approximately the same shape as the mode of free vibration.

The following general conclusions can be drawn with regard to the behavior of a simply supported beam at synchronism. They are also applicable to a continuous structure. These conclusions are approximate and subject to the limitation regarding the position of the periodic force. When synchronism exists between the frequency of the periodic force and a frequency of free vibration the periodic force will excite the mode of free vibration. The amplitude of the mode of free vibration is so large that the total deflection due to a periodic force may be taken as having the shape of the mode of free vibration. This predominance of the mode of free vibration makes the shape of the deflection due to a periodic force independent of the position of the periodic force. The position of the periodic force does, however, affect the amplitude of the mode of free vibration.

The ordinates to the deflection due to a periodic force, therefore, will be directly proportional to the ordinates to the mode of free vibration and the two curves will have the same shape. The ordinates to the deflection curve vary periodically and are in phase, so that they all vary in direct proportion.

The deflection will lag by 90° in phase with the periodic force (times $\sin(2\pi ft - 90°)$ or times $-\cos 2\pi ft$). The deflection due to a periodic force can be taken as described above with small error.

The reciprocal theorem states that the deflection at b, due to a periodic force at a, is equal to the deflection at a, due to a periodic force at b; therefore, the deflection due to a unit periodic force at a, is an influence line for deflection at a.

Since the deflection due to a periodic force has the same shape as a mode of free vibration, the influence line for deflection at any point will also have the same shape as a mode of free vibration. The amplitude of deflection at any point varies with the position of the periodic force in proportion to the ordinates of the mode of free vibration; the shape of the deflection curve is the same as the mode of free vibration and all deflection ordinates change in proportion to the deflection at any point.

A deflection curve, as described above, for which all of the ordinates change in direct proportion, has the properties $\frac{d^2y}{dx^2} \propto y$ at any point.

Since $EI\frac{d^2y}{dx^2} = -M$ the bending moment at any point is proportional to

the deflection at that point. Therefore the influence line for bending moment at any point has the same shape as the influence line for deflection at that same point. The influence line for bending moment at any point may also be taken as having the shape of the mode of free vibration.

In conclusion, the deflection due to a periodic force, the influence line for deflection and the influence line for bending moment may also be taken as having the shape of the mode of free vibration.

A three-span continuous beam exhibits this same behavior as the simply supported beam and is subject to these same conclusions.

The three lowest modes of free vibration in the vicinity of f_1 have been described for a three-span continuous beam. For the frequency of free vibration when $\frac{f}{f_1} = 1$ the shape of the mode of free vibration for each span from left to right is $A \sin \frac{\pi x}{l} \sin 2\pi f_1 t$, $-A \sin \frac{\pi x}{l} \sin 2\pi f_1 t$ ' and $A \sin \frac{\pi x}{l} \sin 2\pi f_1 t$

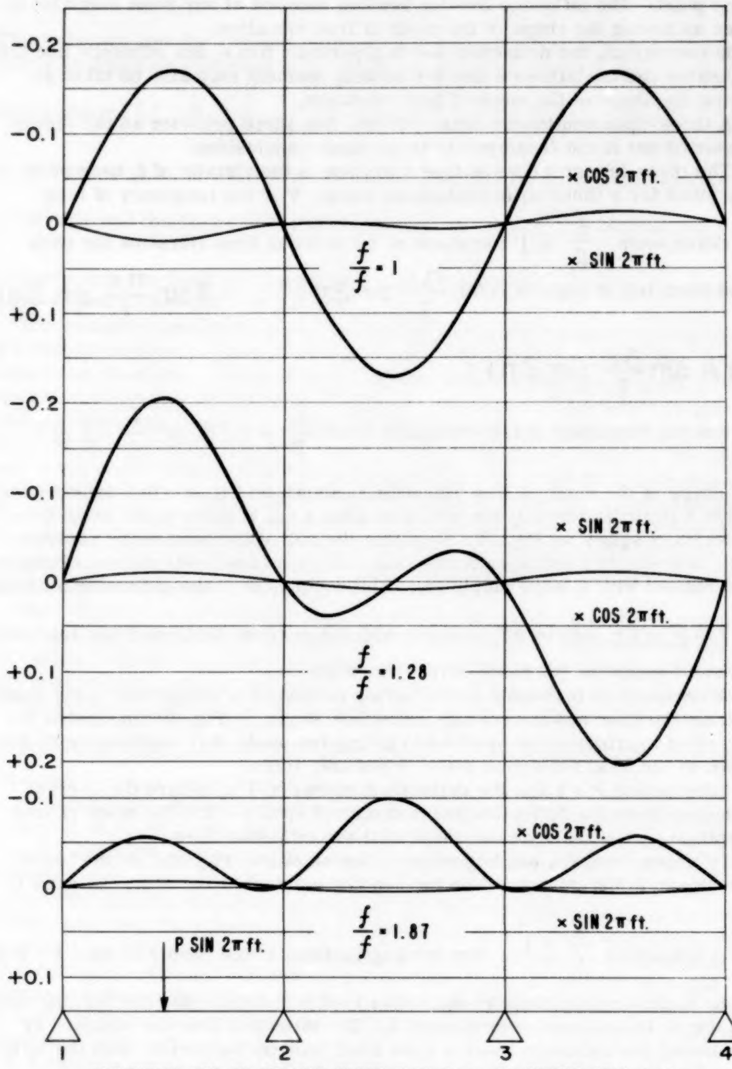
For the frequency of free vibration when $\frac{f}{f_1} = 1.28$ or $\frac{f}{f_1} = 1.87$ the shape of the mode of free vibration is shown on Fig. 3. The deflection due to a periodic force at the center of span 1 - 2 at these same three frequencies is shown on Fig. 10. Note that the only difference in the analysis was the frequency and in each case the corresponding mode of free vibration was excited with a large amplitude. The component of the deflection multiplied by cos $2\pi f t$ lags by 90° in phase with the periodic force and has approximately the same shape as the mode of free vibration.

A comparison between a curve having ordinates in proportion to the mode of free vibration and the correct deflection shown on Fig. 10 illustrates the degree of approximation involved in taking the mode of free vibration as the shape of the total deflection due to a periodic force.

Substituting P = 1 lb., the deflections shown on Fig. 10 are the correct influence lines for deflection at the center of span 1 - 2. The mode of free vibration can be taken as the shape of these influence lines.

Influence lines for bending moment are shown on Fig. 11. At the top of this figure is the influence line for bending moment at the center of span 1- 2 for a frequency $\frac{f}{f_1} = 1$. The bending moment at the center of span 1 - 2 is equal to the bending moment due to the load in the span plus the bending moment due to the moment over support 2. The influence line was obtained by combining the influence line for span 1 - 2, simply supported, with the influence line for bending moment over support 2.

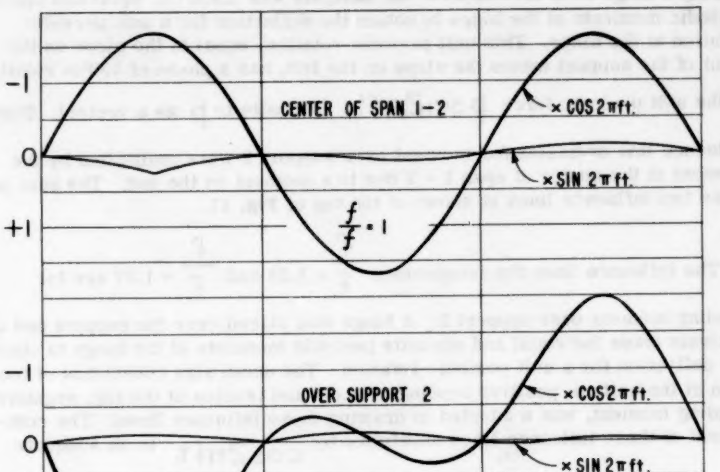
The influence line for bending moment at the center of span 1 - 2, simply supported, was obtained by placing a temporary support and hinge at the center. An analysis was made for equal and opposite periodic moments at the center hinge to obtain the deflection for a unit periodic rotation at the hinge. The reaction at the temporary center support was found and an analysis made



DEFLECTION DUE TO PERIODIC FORCE P SIN $2\pi ft$. AT CENTER SPAN 1-2

$$\text{DEFLECTION} = \frac{Pl^3}{EI}\left(\text{ORD} \times \sin 2\pi ft + \text{ORD} \times \cos 2\pi ft\right)$$

FIGURE 10



INFLUENCE LINES FOR BENDING MOMENT DUE TO PERIODIC FORCE $P \sin 2\pi ft$

BENDING MOMENT = Pl (ORD. × $\sin 2\pi ft$. + ORD. × $\cos 2\pi ft$)

FIGURE 11

for the deflection due to an equal and opposite periodic force at the center of span 1 - 2. The sum of these two deflections is the influence line for bending moment at the center of span 1 - 2, simply supported.

The influence line for bending moment over support 2 was obtained by placing a hinge over the support. An analysis was made for equal and opposite periodic moments at the hinge to obtain the deflection for a unit periodic rotation at the hinge. This unit periodic rotation, equal to the slope on the right of the support minus the slope on the left, has a phase of 180° in relation to the unit periodic force $p \sin 2\pi ft$ (opposite to p as a vector). These influence line ordinates for moment over support 2 were multiplied by the moment at the center of span 1 - 2 due to a moment on the end. The sum of these two influence lines is shown at the top of Fig. 11.

The influence lines for frequencies $\frac{f}{f_1} = 1.28$ and $\frac{f}{f_1} = 1.87$ are for bending moment over support 2. A hinge was placed over the support and an analysis made for equal and opposite periodic moments at the hinge to obtain the deflection for a unit periodic rotation. The usual sign convention of tension at the bottom, positive bending moment and tension at the top, negative bending moment, was a adopted in drawing these influence lines. The component of these influence lines multiplied by $\cos 2\pi ft$ is an adequate representation of the influence line in each case. This component has approximately the same shape as the corresponding mode of free vibration.

This continuous beam behaves in a manner similar to the simply supported beam. The deflection due to a periodic force, the influence line for deflection and the influence line for bending moment may all be taken as having the same shape as the mode of free vibration.

These relations are useful as a guide in the analysis of continuous structures. Since the deflection due to a periodic force will have the shape of the mode of free vibration, then the maximum deflection due to a periodic force will occur at the point of maximum deflection of the mode of free vibration.

Since the influence line for deflection also has the shape of the mode of free vibration, then the position of the periodic force that will cause maximum deflection is at the point of maximum deflection of the mode of free vibration.

Since the deflection due to a periodic force has the same shape as the mode of free vibration, the bending moment diagrams will have the same shape. The maximum bending moment due to a periodic force will occur where the bending moment is maximum for the mode of free vibration.

Since the influence line for bending moment has the same shape as the mode of free vibration, the position of the peroidic force to cause maximum bending moment will be at the point of maximum deflection of the mode of free vibration.

The writer's method of analysis described in Trans. Vol. 118 was used to determine the deflections due to a steady state forced vibration for Figs. 10 and 11. This analysis plus the analysis described in this paper for frequencies and modes of free vibration, for the reciprocal theorem and for influence lines completes the required methods of exact analysis. The similarity which has been shown between the deflection due to a periodic force, the influence

lines for deflection and bending moment, and the mode of free vibration provide a simple and accurate relationship which clarifies the behavior of continuous structures and supplies the understanding necessary to intelligent analysis.

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a. Beginning with "Proceedings-Separate No. 200," published in July, 1953, the papers were printed by the photo-offset method.
b. Presented at the Miami Beach (Fla.) Convention of the Society in June, 1953.
c. Presented at the New York (N.Y.) Convention of the Society in October, 1953.
d. Beginning with "Proceedings-Separate No. 290," published in October, 1953, an automatic distribution of papers was inaugurated, as outlined in "Civil Engineering," June, 1953, page 66.
e. Discussion of several papers, grouped by divisions.
f. Presented at the Atlanta (Ga.) Convention of the Society in February, 1954.
g. Presented at the Atlantic City (N. J.) Convention in June, 1954.

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